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PATH INTEGRAL APPROACH TO BOSONIZATION OF D=2 FIELD THEORI

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Abstract

We suggest a method of bosonizing any D=2 theory. We demonstrate how it works with the examples of the Thirring and the Schwinger models, known results are reproduced. This method, being applied to the Gross-Neveu model, yields nonlinear boson WZW-type theory with additional constraint in the field space. Relation to the nonlinear σ - model is also discussed.

PACS 11.30 Rd Chiral symmetries

1 Introduction

Apart from supersymmetry approach, the equivalence between bose and fermi forms of several D=2 theories is interesting for many reasons. Over the last two decades, when constructing quantized models of corresponding classical fermi theories, the first step was to trasform these theories into bose form [1]. This is the case with the massive Thirring model, which is equivalent to the sine-Gordon [2], the chiral Schwinger model with its bose form [3], D=2 QCD [4] and others [5].

In what sense bose and fermi theories are equivalent? A standard answer [2, 6] is that both these theories reproduce the same quantum currents algebra or, in other words, if both theories give identical correlation functions of bilinear fermion composite operators. This paper proposes another approach: the bose form of a given fermi model is another representation of it and, in fact, is included in this fermi theory. So, we introduce a gauge group valued field Ω, corresponding to anomalous chiral transformation. The anomalous subgroups of the full chiral gauge group G play an exclusive role in our approach because they, and only they can induce the effective boson action. By this method, we are able to obtain the expressions for the nonabelian chiral densities in terms of the boson field Ω(see sec.4 below). This procedure generalizes the similar one for abelian theories(see sec.3 and ref. [9]).

Somewhat analogous approach to bosonization via path integrals was developed by many authors [7]. In these works, by introducing the auxiliary boson fields, the expression for generating functional was obtained in terms of the bose fields after the integrating over fermions. Later, this method became apply to the fermi theories in higher dimensions, for example, to D=4 QCD or SQCD [8].

In the recent years, a more straightforward method of D=2 bosonization was developed for abelian theories [9], where the auxiliary boson field was introduced through the chiral transformation. But in these works, the full recipe of introducing such boson field is absent. Our paper solves the problem of bosonization in a general framework by including a gauge group valued auxiliary field Ω to describe the bose degrees of freedom.

The paper is organized as follows. The method is formulated in sec.2, where we derive the appropriate bosonic action in a general form. In sec.3 we apply this method to several known abelian cases - the massive Thirring and the chiral Schwinger models, after that we proceed (sec. 4) to the non-

abelian case, represented by the U(N) Gross-Neveu model. We shall find that this model is equivalent to the nonlinear chiral WZW - like model with an additional field constraint. This is a new result.Some remarks on the full bosonization problem are scoped in conclusion.

Throughout the paper we use the following notations: $\gamma^0 = \sigma^1, \gamma^1 = i\sigma^2, \gamma^5 = \sigma^3; g^{\mu\nu} = (+,-)$ so that $\gamma_5\gamma_\mu = \epsilon_{\mu\nu}\gamma^\nu$, $\epsilon^{01} = -\epsilon_{01} = 1$. We shall deal with a formulation of a given quantum theory in terms of path integrals. Now we proceed to our bosonization method.

2 A general method

Consider a renormalizable fermion model. This theory is quadratic on fermion fields, or it can be achieved by introducing a set of auxiliary fields. The action is

$$S_0 = \int d^2x(\bar{\psi}\widehat{D}\psi) + S_{boson}.$$

The ψ field belongs to the space of the fundamental representation of the chiral gauge group $U_L(N) * U_R(N)$. The Dirac operator $\widehat{D}$ depends on auxiliary fields and gauge fields A_μ. The generating functional for the Green's functions has the form

$$Z(J) = \int D\bar{\psi}D\psi Db e^{iS(J)}, \tag{1}$$

where $S(J) = S_0 + S_{sources} = \int d^2x(\bar{\psi}\widehat{D}(J)\psi) + S_{boson}, b$ denotes all boson fields in a theory. Here, we have supposed that the sources J are introduced for fermion bilinears such as $\bar{\psi}\psi, \bar{\psi}\gamma_\mu T^a\psi$, $\bar{\psi}\gamma_\mu\gamma_5 T^a\psi$ and others. That is because we are interested in the correlation functions for currents in a theory, for instance, the quantum currents algebra.

Consider now a new functional $Z^1(J)$, which has the same Green's functions for currents operators in terms of $\bar{\psi}$, ψ fields which $Z(J)$ has:

$$Z^1(J) = \int D\Omega Z(J), \tag{2}$$

$D\Omega$ is a measure on the gauge group G, and Ω is a function of $x : \Omega = \Omega(x) \in G$. We have introduced a new field Ω according to a given representation

of a fermion wave function ψ, Ω is an element of the chiral gauge group $G = U_L(N) * U_R(N)$.So the quantum numbers of Ω are just the same as of the corresponding gauge group parameters.

After substitutions

$$\psi \to \Omega\psi, \psi^+ \to \psi^+\Omega^+,$$

we obtain:

$$\begin{gathered} D\bar{\psi}D\psi \to e^{i\alpha_1(A,\Omega)} D\bar{\psi}D\psi, \\ \widehat{D}(J) \to \widehat{D}(J,\Omega) = \tilde{\Omega}\widehat{D}(J)\Omega, \\ \tilde{\Omega} = \gamma_0\Omega^+\gamma_0, \end{gathered} \tag{3}$$

here $\alpha_1(A,\Omega)$ is a 1-cocycle on the gauge group [10] (if theory has no gauge fields, it is reduced to $\alpha_1(0,\Omega)$). In fact α_1 is the WZW action, including a topological part for the nonabelian groups [6, 11]. From the general property of the 1-cocycle

$$\alpha_1(A^\Omega, g) - \alpha_1(A, \Omega g) + \alpha_1(A,\Omega) = 0,$$

we can deduce useful identities $\alpha_1(A^\Omega,\ \Omega^{-1}) = -\alpha_1(A,\Omega)$; $\alpha_1(A^\Omega, 1) = 0$. Now (2) has a form

$$Z^1(J) = \int D\Omega D b D\bar{\psi} D\psi \exp(iS(J,\Omega) + iS_{boson} + i\alpha_1(A,\Omega)),$$

where

$$S(J,\Omega) = \int d^2x(\bar{\psi}\widehat{D}(J,\Omega)\psi),$$

and $\widehat{D}(J,\Omega)$ is given by (3).

After integrating over the fermi fields we finally have a bosonic form of the theory (1):

$$Z^1(J) = \int D\Omega D b \exp(\frac{i}{2}\mathrm{Tr}\log\widehat{D}\widehat{D}^+(J,\Omega) + iS_{boson} + i\alpha_1(A,\Omega)). \tag{4}$$

So for any given fermion theory (1) we are able to construct a bose theory (4), both these theories have the same correlation functions for currents and hence, are equivalent quantum mechanically. This is the main result of this paper. Take into account that it is necessary to do the Wick rotation to define $\mathrm{Tr}\log\widehat{D}\widehat{D}^+$ in (4) on euclidian compactified space T^2.

Note that instead of (2), one can consider a more general functional $Z^1_\rho(J) = \int D\Omega\rho(\Omega)Z(J)$, where $\rho(\Omega)$ is an arbitrary weight functional $\rho : G \to \mathbf{R}$, but we don't discuss here an ambiguity of this kind.

Now, turn to some D=2 fermion models and their bosonized versions.

3 Two abelian models

Starting from an abelian case, consider the Thirring model [2]

$$S(J) = \int d^2x(\bar\psi(i\hat\partial - m)\psi - \frac{1}{2}g\bar\psi\gamma_\mu\psi\bar\psi\gamma^\mu\psi + J_\mu\bar\psi\gamma^\mu\psi) =$$

$$\int d^2x(\bar\psi\widehat{D}(J)\psi) + S_{boson}, \tag{5}$$

$$\widehat{D}(J) = i\hat\partial - g\hat{a} + \hat{J} - m,$$

$$S_{boson} = \frac{g}{2}\int d^2x a_\mu a^\mu.$$

In this case we can't choose the group element $\Omega = e^{i\alpha(x)}$ because $\alpha(x)$ here is a phase of the wave function $\psi(x)$, and it is unobservable. Instead of it, we use a fact, that in the massless limit $m \to 0$, the Thirring model has a symmetry $U_L(1) * U_R(1)$. So, we are able to choose Ω as an element of the coset $U_L(1) * U_R(1)/U(1)$, $\Omega = e^{i\gamma_5\alpha(x)}$ and

$$\widehat{D}(J,\Omega) = i\hat\partial - \gamma_5(i\hat\partial\alpha) - g\hat{a} + \hat{J} - me^{2i\gamma_5\alpha},$$

due to (3). Then, we are going to calculate $\mathrm{Tr}\log\widehat{D}\widehat{D}^+(J,\Omega)$ using, for example, the proper - time regularization method for elliptic operator $\widehat{D}\widehat{D}^+(J,\Omega)$:

$$\mathrm{Tr}\log\widehat{D}\widehat{D}^+(J,\Omega) = -\mathrm{tr}\int_\epsilon^\infty \frac{dt}{t}\mathrm{Tr}e^{-t\widehat{D}\widehat{D}^+(J,\Omega)} =$$

$$-\int_\epsilon^\infty \frac{dt}{t}\mathrm{tr}\int d^2x\frac{d^2k}{(2\pi)^2}e^{ikx}e^{-t\widehat{D}\widehat{D}^+(J,\Omega)}e^{-ikx} =$$

$$-\int_\epsilon^\infty \frac{dt}{t}\mathrm{tr}\int d^2x\frac{d^2k}{(2\pi)^2}\exp(-te^{ikx}\widehat{D}\widehat{D}^+(J,\Omega)e^{-ikx}) =$$

$$-\int_{\epsilon}^{\infty}\frac{dt}{t}\mathrm{tr}\int d^2x\frac{d^2k}{(2\pi)^2}e^{-tk^2}\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}(B_\mu k^\mu + C)^n =$$

$$\frac{A}{16\pi}\mathrm{tr}\int d^2x(4C - B_\mu B^\mu) - \frac{1}{16\pi}\mathrm{tr}\int d^2x B^\mu B^\mu + O(\epsilon), \qquad (6)$$

where $A \to \infty$ as $\epsilon \to 0$, tr-trace over γ-matrices;

$$C = \widehat{D}\widehat{D}^+(J,\Omega), B_\mu = \gamma_\mu\widehat{D}^+ + \widehat{D}\gamma_\mu,$$

so that $\mathrm{tr}(4C - B_\mu B^\mu) = -2\mathrm{tr}\gamma_\mu\widehat{D}\gamma^\mu\widehat{D}^+ = -4m^2\mathrm{tr}e^{4i\gamma_5\alpha(x)} = -8m^2\cos 4\alpha(x)$, because in D=2 for any vector $P_\mu : \gamma_\mu\widehat{P}\gamma^\mu = 0$. Also in (6) we have used an operator identity $e^A e^B e^{-A} = \exp(e^A B e^{-A})$. The only term remained in (6) is

$$\mathrm{tr}B_\mu B^\mu = \mathrm{tr}(2\gamma_\mu\widehat{D}\gamma^\mu\widehat{D}^+) =$$

$$8(\partial_\mu\alpha\partial^\mu\alpha + 2g\epsilon^{\mu\nu}(\partial_\nu\alpha)a_\mu + g^2 a_\mu a^\mu + 2\epsilon^{\mu\nu}(\partial_\nu)J_\mu + J_\mu J^\mu + m^2\cos 4\alpha).$$

After substitution of (6) in (4) and integrating over a_μ fields in (4), one exactly gets sine-Gordon theory (after rescaling $\alpha \to \sqrt{\pi}\alpha$). Note, that the source J_μ in (5) is coupled to the vector current in a fermionic form $\bar{\psi}\gamma^\mu\psi$, while in boson theory it is coupled to the same current in terms of bosonic field $\alpha(x)$, so we immediatly have

$$\bar{\psi}\gamma^\mu\psi = \frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\alpha$$

as expected [2, 6]. Corresponding action turns out to be

$$S(J) = \int d^2x(\frac{Z_1}{2}\partial_\mu\alpha\partial^\mu\alpha + \frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}(\partial_\nu\alpha)J_\mu + m^2 Z_m\cos 4\alpha),$$

$$Z_1 = (1 + \frac{g}{\pi})^{-1}, Z_m = \frac{A+1}{2\pi}.$$

The method of bosonization nonabelian Thirring model at 1/N expansion was also considered in literature [12]. The second interesting example is the chiral Schwinger model, widely discussed by many authors [1, 3], subject to quantization of an anomaly theory. This model is described by the action

$$S_0 = \int d^2x(\bar{\psi}\widehat{D}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}), \qquad (7)$$

$$\widehat{D} = i\hat{\partial} + e\sqrt{\pi}\hat{A}(1-\gamma^5) = i\hat{\partial} + 2e\sqrt{\pi}\hat{A}P_-.$$

The fermions belong to the representation space of the abelian group $G = U_L(1) * U_R(1)$.Choosing $\Omega = \exp(i\sqrt{\pi}(P_-\alpha + P_+\beta)) \in G$; $P_\pm = \frac{1\pm\gamma_5}{2}$ - the chiral projection operators such as $P_\pm\gamma_\mu = \gamma_\mu P_\pm$, we obtain

$$\widehat{D}(\Omega) = \tilde{\Omega}\widehat{D}\Omega = i\hat{\partial} + \hat{a}P_- + \hat{b}P_+,$$

where

$$\hat{a} = \sqrt{\pi}(2e\hat{A} - \hat{\partial}\alpha), \hat{b} = -\sqrt{\pi}\hat{\partial}\beta,$$

and in a way, similar to the eq.(6), we get

$$\mathrm{Tr}\log\widehat{D}\widehat{D}^+(\Omega) = -\frac{1}{16\pi}\mathrm{tr}\int d^2x B_\mu B^\mu + O(\epsilon),$$

with vanishing term $\mathrm{tr}(4C - B_\mu B^\mu)$, and

$$\mathrm{tr}B_\mu B^\mu = 4\mathrm{tr}\widehat{D}\widehat{D}^+(\Omega) = 16\pi(\partial_\mu\alpha\partial^\mu\alpha + 2eA_\mu(g^{\mu\nu}+\epsilon^{\mu\nu})\partial_\nu\alpha + 2e^2A_\mu A^\mu), \quad (8)$$

where we have taken $\alpha = -\beta$. In deriving the eq.(8), it is necessary to continue γ_5 matrix properly in $\mathbf{R^2}$, because it changes its own hermitian property $\gamma_{5E}^+ = -\gamma_{5E}$, and $\gamma_{5E}^2 = -1$, so one has an unusual algebra of the projection operators: $P_\pm^E P_\pm^E = \pm\gamma_{5E}, P_\pm^E P_\mp^E = 1$ instead of $P_\pm P_\pm = P_\pm, P_\pm P_\mp = 0$. After substitution of (8) into (4) , one gets the bosonized chiral Schwinger model with a WZW term [3].

4 Nonabelian model

Obviously, the nonabelian models are the most interesting since they lead to nonlinear boson theories which can be quantized by the Faddeev-Jackiw method [13] for constraint theories. In this section we are dealing with the U(N) Gross-Neveu model, where fermions form a space of the fundamental representation of the group $G = U(N)$. We shall see that this model leads to somewhat unusual boson theory.

To begin with, let us consider a free-fermion model with the sources $J_{\mu L}, J_{\mu R}$ for the left and the right currents respectively, coupled as follows:

$2\bar{\psi}\widehat{J}_L P_- \psi$ and $2\bar{\psi}\widehat{J}_R P_+ \psi$; $\widehat{J}_{L,R} = J^a_{\mu L,R} T^a \gamma^\mu$. Then,

$$S(J) = \int d^2 x \bar{\psi}_i \widehat{D}^{ij}(J) \psi_j =$$

$$\int d^2 x \bar{\psi} \widehat{D}(J) \psi, \tag{9}$$

$$\widehat{D}(J) = i\widehat{\partial} + 2\widehat{J}_L P_- + 2\widehat{J}_R P_+,$$

and the element of the chiral symmetry group G is choosen to be $\Omega^1 = P_-\Omega + P_+\Omega^+ \in G = U_L(N) * U_R(N)$. So, for the matrix operator $\widehat{D}(J,\Omega) = \tilde{\Omega}^1 \widehat{D}(J) \Omega^1$ one finds

$$\mathrm{tr} B_\mu B^\mu = 16 \mathrm{tr}_u ((\partial_\mu \Omega)(\partial^\mu \Omega^+) + \frac{1}{2}\epsilon^{\mu\nu}(\partial_\mu \Omega)(\partial_\nu \Omega)\Omega^{+^2} -$$

$$\frac{1}{2}\epsilon^{\mu\nu}(\partial_\mu \Omega^+)(\partial_\nu \Omega^+)\Omega^2 - \frac{i}{2}\epsilon^{\mu\nu} J_{\mu L}\Omega(\partial_\nu \Omega^+) + \frac{i}{2}\epsilon^{\mu\nu} J_{\mu R}(\partial_\nu \Omega)\Omega^+), \tag{10}$$

where tr_u-a trace over the U(N) group, while quadratic in sources terms are omitted since they have not contributed to the correlation functions in (4). The substitution of (10) into (4) gives the nonlinear sigma model with a WZW term and a topological term related to $\pi_2(S^2)$. Indeed, choosing a form of the element of U(2), for example, is $\Omega = n_0 I + i\sigma\mathbf{n}$, unitarity condition $\Omega\Omega^+ = 1$ transforms into $n_i^2 = 1$, i=0,1,2,3, and

$$\mathrm{tr}_u (\partial_\mu \Omega)(\partial^\mu \Omega^+) = 2\partial_\mu n_i \partial^\mu n_i,$$

$$\mathrm{tr}_u \epsilon^{\mu\nu}(\partial_\mu \Omega)(\partial_\nu \Omega)\Omega^{+^2} = 4 n_0 \epsilon^{\mu\nu}\epsilon_{abc}\partial_\mu n_a \partial_\nu n_b n_c.$$

So one gets the O(3) nonlinear sigma model, if $n_0 = const$ [14].

Corresponding action turns out to be

$$S_{free} = \frac{1}{\pi}\int d^2 x (\frac{1}{2}\partial_\mu n_a \partial^\mu n_a + const\epsilon^{\mu\nu}\epsilon_{abc}\partial_\mu n_a \partial_\nu n_b n_c) + S_{WZW}. \tag{11}$$

The second term is a degree of the map $n_a(x) : S^2 \to S^2$. Note that from (10) and (4) one has the expressions for the currents in bosonized theory such as

$$\bar{\psi}\gamma^\mu (1+\gamma_5) T^a \psi = \frac{i}{2}\epsilon^{\mu\nu}\mathrm{tr}_u (T^a \partial_\nu \Omega \Omega^+),$$

$$\bar{\psi}\gamma^{\mu}(1-\gamma_5)T^a\psi = -\frac{i}{2}\epsilon^{\mu\nu}\mathrm{tr}_u(T^a\Omega\partial_\nu\Omega^+).$$

Now add the Gross-Neveu interaction term to a free theory (9):

$$S_{int} = \frac{g^2}{2}\int d^2x(\sum_{i=1}^{N}\bar{\psi}_i\psi_i)^2 = g\int d^2x\phi\bar{\psi}\psi - \frac{1}{2}\int d^2x\phi^2.$$

It contributes to (11)

$$4g^2\phi^2\mathrm{tr}_u(\Omega^4+\Omega^{+^4}), \tag{12}$$

so, for the boson theory one finds

$$S(\Omega) = S_{free} + \frac{g^2}{4\pi}\int d^2x\phi^2\mathrm{tr}_u(\Omega^4+\Omega^{+^4}) - \frac{1}{2}\int d^2x\phi^2.$$

After the integrating over ϕ field one obtains the nonlinear sigma model action (11) with additional to $\Omega\Omega^+ = 1$ constraint from (12): $\mathrm{tr}_u(\Omega+\Omega^+)^4 - 2\mathrm{tr}_u(\Omega+\Omega^+)^2 = const$ as for $\Omega \in U(N)$ one has $\Omega^4 + \Omega^{+^4} = (\Omega+\Omega^+)^4 - 2(\Omega+\Omega^+)^2 - 2$. This reduces the phase space of the theory (11), giving a new result.

5 Conclusion.

In this paper, for any given fermion theory in D=2, we have constructed the equivalent quantum mechanically bose theory, which is described by the action (4). Physically it means, that theory (4)is a nothing but the effective one for the fields, extracted from the phase space of the fermion wave functions in some representation of the internal symmetry group G. By this method, we have obtained the bose theory for the Gross-Neveu model, which appeares to be a nonlinear WZW - type model with a topological term and additional constraint in the phase space. Note, that the appearence of the spin structure in such a model was confirmed by the independent method [15].

Many other applications of this explicit bosonization procedure can be considered. For example, if one enlarges the internal symmetry group to the

conformal group, it can be possible to obtain the D=2 induced gravity. It can be used also in D=4 theories for deducing the low- energy effective actions.

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